TOPOLOGY - III, EXERCISE SHEET 4

This exercise sheet is a revision on some important notions from homological algebra. Throughout the sheet the word "Abelian Group" can be replaced with the words "$k-$vector space" for a field k or "$R-$module" for a ring R and the results mentioned continue to hold.

Definition 4.1: A chain complex $(C_\bullet, \partial)$ is a collection $\{C_i\}_i \in \mathbb{Z}$ of Abelian groups along with a collection of homomorphisms $\partial_i : C_i \to C_{i-1}$, called boundary maps, such that for all $n \in \mathbb{Z}$ we have that $\partial_{i-1} \circ \partial_i = 0$. We define the groups $H_i(C_\bullet)$, called the homology groups, to be the quotient $\frac{ker\, \partial_i}{Im\, \partial_{i+1}}$.

A morphism of chain complexes $f : (C_\bullet, \partial^C) \to (B_\bullet, \partial^B)$ is a collection of group homomorphisms $\{f_i : C_i \to B_i\}_{i \in \mathbb{Z}}$ such that $\partial_i^B \circ f_i = f_{i-1} \circ \partial_i^C$ for all $i \in \mathbb{Z}$.

When no confusion is likely to arise, we drop the boundary maps from the notation and denote a chain complex by $C_\bullet$ instead of $(C_\bullet, \partial)$.

Exercise 1.
(1) Show that a morphism of chain complexes $f : C_\bullet \to B_\bullet$ naturally induces group homomorphisms $H_i(f) : H_i(C_\bullet) \to H_i(B_\bullet)$ for all $i \in \mathbb{Z}$.
(2) Show that the set of homomorphisms $Hom(C_\bullet, B_\bullet)$ has a natural structure of an abelian group.
(3) Let $f : C_\bullet \to B_\bullet$ be a morphism of chain complexes show that the boundary maps on $C_\bullet$ restrict to maps $\partial_i : Ker\, f_i \to Ker\, f_{i+1}$. Hence we can define the kernel of f to be the complex $Ker\, f_\bullet$. Similarly show that we can define the image and co-kernel for a morphism of complexes. Therefore observe that it also makes sense to talk about exact sequences of chain complexes.

Exercise 2. Given a topological space X, recall that one can form the complex of singular chains $(C_\bullet(X), \partial)$, a complex of free abelian groups. Also recall that $f : X \to Y$, a continuous map of topological spaces induces a morphism $f_*^n : C_n(X) \to C_n(Y)$ for all n.

(1) Show that the identity map $X \to X$ induces the identity homomorphism $C_i(X) \to C_i(X)$ for all i.
(2) Given continuous maps of topological spaces

$$X \xrightarrow{g} Y \xrightarrow{f} Z.$$

Show that $(f^i \circ g^i)_* = f_*^i \circ g_*^i : C_i(X) \to C_i(Z)$ for all $i \in \mathbb{Z}$.
(3) Let $f : X \to Y$ be a continuous map. Prove that f_* defines a morphism of chain complexes $C_\bullet(X) \to C_\bullet(Y)$. Conclude that f induces a map $f_* : H_i(X) \to H_i(Y)$ for all i.

Exercise 3. *Homotopy of morphisms of chain complexes.*

Definition 4.2: Let $f : C_\bullet \to B_\bullet$ be a morphism of chain complexes, we say the f is null-homotopic or $f \sim 0$ if there exists a collection of group homomorphisms $\{h_i : C_i \to B_{i+1}\}_{i\in\mathbb{Z}}$ called a homotopy such that $f_i = \partial_{i+1} \circ h_i + h_{i-1} \circ \partial_i$. We say that $f \sim g$ if $f - g \sim 0$.

(1) Show that if $f \sim g$ then the induced maps on homology $H_i(f) : H_i(C_\bullet) \to H_i(B_\bullet)$ and $H_i(g) : H_i(C_\bullet) \to H_i(B_\bullet)$ are the same for all $i \in \mathbb{Z}$.
(2) Show that $\sim$ is an equivalence relation. Show that null-homotopic maps of complexes form a subgroup of $Hom(C_\bullet, B_\bullet)$ and hence observe that the equivalence classes induced by the relation $\sim$ are exactly the cosets of the subgroup of null-homotopic maps. In particular chain complex morphisms up to homotopy form an abelian group.

Exercise 4. *Long exact sequence in Homology.*
The aim of this exercise is to construct the long exact sequence in homology arising from the short exact sequence of chain complexes. That is, let

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$$

be a short exact sequence of chain complexes of abelian groups. Then we have a long exact sequence of homology groups:

$$... \xrightarrow{\delta_{i+1}} H_i(A_\bullet) \xrightarrow{H_i(f)} H_i(B_\bullet) \xrightarrow{H_i(g)} H_i(C_\bullet) \xrightarrow{\delta_i} H_{i-1}(A_\bullet) \xrightarrow{H_{i-1}(f)} ...$$

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(1) Constructing $\delta_i : H_i(C_\bullet) \to H_{i-1}(A_\bullet)$:

   (a) Let $x \in H_i(C_\bullet)$. Choose a lift $\overline{x} \in C_i$ of x.
   (b) Let $y \in B_i$ be such that $g_i(y) = \overline{x}$. Why does such a y exist?
   (c) Consider the element $\overline{y} := \partial_i^B(y) \in B_{i-1}$. Let $z \in A_{i-1}$ be such that $f_{i-1}(z) = \overline{y}$. Why does such a z exist?
   (d) Show that $z \in Ker\,\partial_{i-1}^A$.
   (e) Let $\overline{z}$ be class of z in $H_{i-1}(A_\bullet)$. We define $\delta_i : H_i(C_\bullet) \to H_{i-1}(A_\bullet)$, $\overline{x} \mapsto \overline{z}$. Show that δ_i is a well-defined group homomorphism. That is, $\overline{z}$ is independent of the choices made in the above steps.

(2) Show that with the δ_i constructed in (1), the sequence of homology groups above is an exact sequence.